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ABSTRACT

There has been considerable interest in mathematical understanding. Both those attempting to build, and those questioning the possibility of building intelligent artificial tutoring systems, struggle with the notions of mathematical understanding. The purpose of this essay is to show a transcendently recursive theory of mathematical understanding which is based on the consideration of understanding as a nonlinear process of growth. This theory of growth has a built-in dynamic of folding back to move out, but also shows understanding as an organizing process to entail action, perception of action, conception of percepts, conception of concepts, conception of forms (or classes) and logical explanation in an embedded manner. The theory and its elements, and the properties and applications of the theory are discussed. (CW)

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A recursive theory for mathematical understanding - some elements and implications.*

A paper prepared for the AERA Boston Meeting - April, 1990

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There is considerable current interest in understanding. This can be seen in the context of the NCTM Standards for Curriculum and Evaluation or in the debate surrounding the implementation of the new National Curriculum in Britain. In a completely different area of educational concern, both those attempting to build, and those questioning the possibility of building intelligent artificial tutoring systems, struggle with the notions of understanding. Many people are using the word, but is the meaning they attach to it clear or sufficient for their tasks?

There also has been considerable interest in the mathematics education literature over the last 20 years in mathematical understanding (Schroder, 1987). Writers have tried to capture understanding using many categories such as instrumental, relational, concrete, symbolic and so on. Others have suggested that understanding could be thought of as a vector of values in such categories (Tall, 1978). Pirie (1988) called into question use of categories or vectors of categories in characterising the understanding as actually seen in young persons. She observed understanding as a whole dynamic process and not as a single or multi-valued acquisition, nor as a linear combination of such values or knowledge categories.

I. The theory and its elements

A central question of our on-going work together over the past two years has been to try to more fully characterise mathematical understanding as a non-linear, dynamic growing process. Our purpose here is to present our recursive theory of mathematical understanding; its elements and properties. We will try to show some of its implications for research in the study of mathematical behaviour, and in addition look at its use in teaching mathematics.

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Our work has been informed by literature from many sources and fields. Our theory represents an attempt to elaborate the constructivist definition of understanding as a process . In 1987 von Glasersfeld wrote: "The experiencing organism now turns into a builder of cognitive structures intended to solve such problems as the organism perceives or conceives . . . among which is the never ending problem of consistent organisation[of such structures] that we call understanding".

Our work was also stimulated by the ontological work of Maturana and Varela (1980, 1987). They see persons as self-referencing, capable of making distinctions in their own physical and mental behaviour and in their interactions with the environment and others in it. Knowing for them is considered to be the ability to take effective action in the eyes of an observer. Such knowing grows recursively using distinctions made on previous knowing. Following from that work, Tomm (1989) envisions knowing and its growth to be situated, not in the text book or library,but in the interaction with "the other" such as a teacher or fellow student. For him knowing grows recursively through making distinctions from the level of actions, to a level of linguistic actions, to languaging and to a level of self-conscious thought and beyond.

Our theory uses recursion not only in Tomm's sense which resonates with Piaget's notion of reflective abstraction, but in a broader way. With Vitale (1989) we see recursion as a way of observing a phenomenon (in this case mathematical understanding) involving embeddedness, self similarity and level stepping. We extend this notion of level-stepping by using an idea from Margenau(1987). He emphasised that if one looked at the historical growth of ideas in science one saw <u>transcendence</u>. That is, new ideas do not simply embroider or elaborate on the old. There is no reductionism - the new concept is not reducible to the old. There is <u>transcendence</u> with <u>compatibility</u>.

What does our theory actually look like ? After many recursions in our own understanding, we have come up with a "picture" which is a portrayal of the theory. Before giving that picture we will try to describe and exemplify the levels which it entails. But the reader should be aware of the fact that the levels by themselves do not constitute understanding but are but named parts of a whole phenomenon. This "picture" is found in Figure 1 below.

We will define and illustrate the eight levels of our theory by tracing through the building of an understanding of the concept of fractions.

The process of coming to know starts at a level we call primative doing. Action at this level can involve physical objects, figures, graphics or symbols. Primative here does not imply low level mathematics but rather the starting place for growth for any particular mathematical understanding. It is what the observer, the teacher or the researcher, assumes that the person doing the understanding can do initially. For example if one wanted to observe the growth in understanding of 10 or 11 year old children studying fractions one might assume they could divide up continuous things - such as pizzas - or sets of discrete things - such as sweets - and that they could use at least some fraction words and symbols descriptively.

The first recursion or level change occurs when the learner begins to form images through specific doing. This represents a genuine level change in that the learner is asked to make distinctions in her previous abilities and use them perhaps under new circumstances or to new ends.
We call this level image making . For fractions children would successfully engage in solving sharing problems and use fraction language to record their actions. For instance: sharing three pizzas between four people and writing or saying "three quarters of a pizza each."

At the next level these singular action-tied images are replaced by a form for the images. It is this image having which frees a person's mathematics from the need for particular actions or examples. This image is a mental object which can now be used in mathematical knowing.
At this level children might say "fractions show the amounts that come from dividing up" or, more interestingly, "different ways of dividing up into fraction pieces can show the same amount."

At the next level, property noticing, these images can now be examined for specific or relevant properties. This involves noting distinctions, combinations or connections between images. Now children might notice that equivalent fractions can be generated by multiplying numerator and denominator by the same value, and can record this process by writing for example "2/3 = 4/6 = 6/9 =..." This allows them to create and talk about fractions that might not be in their direct experience for example 411/6 hours.

The next level of understanding, formalising, entails thinking consciously about the noticed properties and abstacting common qualities. It is at this level that full mathematical definitions could occur. The person now has a

new class-like mental object. Children at this level can see things in terms of "for all fractions" and know that a/b can represent any fraction, although they might not be using algebraic language. Fractions now have a new status. They have become a set of numbers not actions.

One is now in a position to observe ones own formal thought structures and consistently organise them. One is aware of being aware and can see the consequences of ones thoughts and not merely of ones actions. At this level a child might make the *observation* "there can be no smallest fractional number." This would contrast with an earlier *property* " I can make a smaller fraction by adding 1 to the denominator. e.g. 1 over 9 is smaller than 1 over 8." Both would contrast with the even more general but concrete *image* "more parts means smaller parts." (This latter remark is suggestive of the fact that outer level knowing is not necessarily more general knowing, although it might be.)

For fuller understanding one must now be able to answer why these formal observations must be true. This explaining calls for an awareness of associations and sequence within ones observations and their interdependence. This might be setting ones thinking within an axiomatic structure. We call this level structuring and question whether this level is in fact peculiar to mathematics. At this level addition of fractions is not combining physical quantities - which would be *image making*. It is not an action oriented consequence of equivalence - which would occur at the *property noticing* level. It is not even a *formalised* algorithm. Addition of fractions is simply a logical property following from other logical properties.

The outer most level of recursion of the eight in our model we call the level of inventing. We have had some trouble with this label as we do not wish to imply that children do not "invent" at other levels. But we do wish to point to a special, new activity. Within a given topic a person at this level has a full structured understanding. They may therefore be able to break away from the preconceptions which brought about this understanding and create new questions which might grow into a totally new concept. An example of this might arise as follows:
At the *structuring* level a person can see the rationals as a set of numbers of the form a/b. This set of numbers is also seen to be a quotient field. This person might now *invent* by asking: "what might numbers of the form a/b/c be like?" (It was a question like this which stimulated Hamilton to think about and finally develop quaternions from having the *structured* understanding of the complex numbers.)

We have just given the eight levels from primitive doing to inventing which we think serve as the constituent parts of understanding. However, we warned against focussing on the levels of knowing as effective action by themselves. Remember that we are trying to observe understanding as a dynamic and organising process. We envision these eight levels fitting into a structure as shown in this diagram:

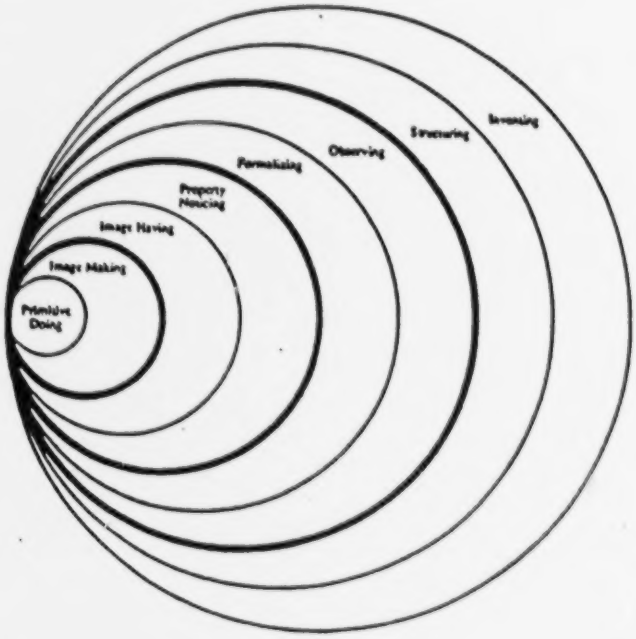

Figure 1: An Eight Level Model for the Theory

One obvious consequence of this model is that, outer levels grow recursively from the inner ones, but knowing at an outer level allows for and indeed retains the inner levels. Outer levels embed and enfold the inner ones. In fact this a relativity theory of understanding and therefore a particular feature of primitive doing is that observers can consider what they wish as the focus at this level. For example, one could observe a person at the inventing level as having their entire previous understanding as a new primitive doing. (At this point we contemplated creating a figure to illustrate this, but what we would in fact need would be a video tape to demonstrate the dynamic nature of the operation!) A main concequence of this line of thinking is that, to an observer, understanding has a fractal-like quality. One could look at

the understanding of a person "inside" primitive doing and observe the same levelled structure.

A second critical element of our theory, observable in the model, is illustrated by the bold rings. We call these the "don't need" boundaries. When a person has an image of a mathematical idea, she does not need action or the specific instances of image making....

...when I have a picture for fractions I can stop cutting up pieces of paper.....

The next "don't need" boundary occurs between property-noticing and formalising. A person who has a formal mathematical idea does not need an image....

...I can think of fractions as things like a/b with no reference to partitioning....

The third "don't need" boundary occurs between observing and structuring. A person with a mathematical structure does not need the meaning brought to it by any of the inner levels.....

...I can add, divide etc without any reference to what a fraction really represents.

It is this feature of the "don't need" boundaries which is an indication of the transcendent nature of the model.

The discussion so far might lead you to focus your attention on the levels and cause you to think that they are critical. They are necessary and structurally important. But a more crucial feature is " **folding back**".

When faced with a problem or question at any level, which is not immediately solvable, one might be prompted to fold back to an inner level in order to extend one's current, inadequate understanding.

But this returned to inner level action is not identical to the original inner level action - it is now informed and shaped by outer level interests. This inner level action is part of a recursive reconstruction of knowledge, necessary to further build outer level knowing. There is no linear, outwards progression in understanding. Let us consider an example of this folding back process.

In the case of fractions I might be at the stage of formalised knowing and be asked " how might one combine these things? and one might consider an example, 1/2+2/3= ?" I could be told the rule: "find a common denominator, and cross multiply to find the numerators". This would give me an action but not necessarily understanding. Or I might think: "what are these things - fractions? They came from cutting things up." I could form the image-"a" below- or i could actually do the cutting up. I can now use my understanding of equivalence which allows me to divide (or cut) them up further - into 1/6ths(see "b")-

a)

b)

and count the parts (or reassemble the pieces) to give me 7/6ths or one whole one and an extra 1/6th.
Thus from the level of formalisation I folded back to image having (or even image making) to make some sense of the operation at the formal level. In fact, I enlarged my image by adding the idea of being able to combine, as well as divide up, fractions. After that I reused my idea of equivalence(at the image having or property noticing level) now in the aid of addition. It would have been of no value to me to simply re-call what I did previously. I needed to re-member my images to form a new way of looking. This folding back enabled a reconstruction of my inner level knowing as a foundation for outer level understanding. This distinction between re-call and re-member is one of the things which we think characterises the difference between artificial intellegence and human understanding. The diagram below illustrates this particular folding back on the ringed model.(Figure 2)

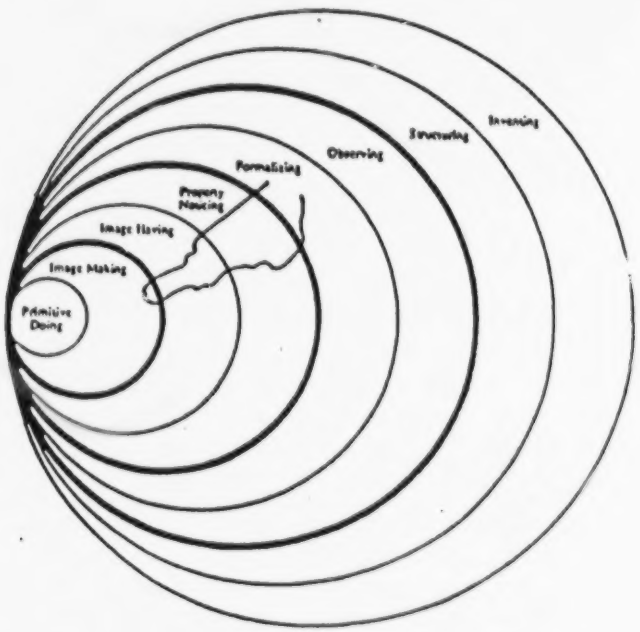

Figure 2: Folding Back

II. Properties and Applications

The three elements of the theory - embeddedness, "don't need" boundaries and folding back - give us power to portray the dynamical process which we are taking to be understanding.

One other element which also contributes substantially to this process is the nature of the levels themselves. We take every level beyond primitive doing as composed of a complementarity of form and process. This extended model is illustrated in Figure 3 below.

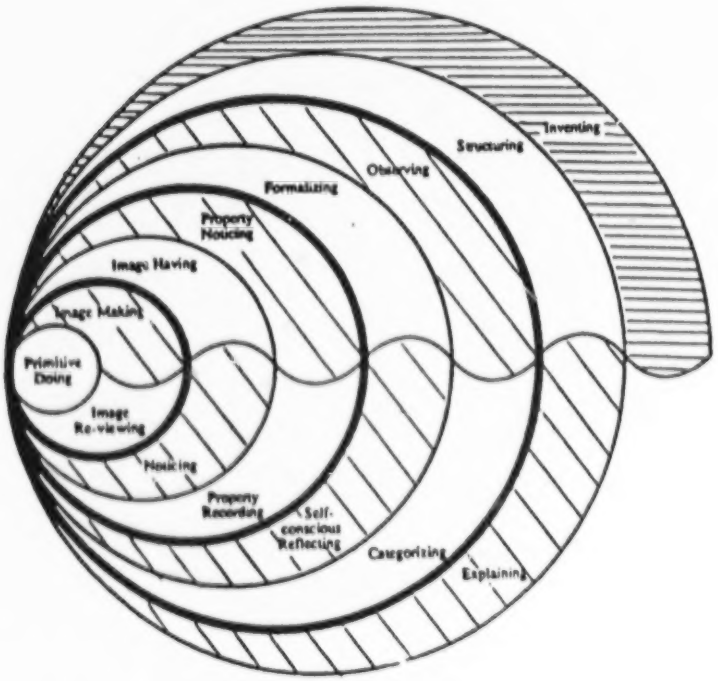

Figure 3 The Within Level Complementarities

The first recursive level beyond primitive doing is composed of the **process of image making** and the **form of image re-viewing.** The next level has the **process of noticing(or image seeing)** complemented with the formal aspect of image-having.The next level has the **process of property noticing** associated with the **formal aspect of property recording.** The next level has the **process of self-conscious reflection** paired with **formalising.** The next level combines **observing** with the **formal activity of categorising.** And finally at an outer level the **process of explaining** leads to the **formal action of structuring.**

It is beyond the scope of this paper to give careful definitions to all of these processes and forms, but some brief observations are in order. First one can notice that the rings in the model in Figure 1 are named alternately by the process and the form of the model. This was done to suggest the alternating conceptual emphases of the levels.

But within any level we are saying that both aspects are necessary to complete a level and hence lead to growth. For example in image making with fractions one might actually create a partition to solve a sharing problem. Image re-viewing at that level simply suggests that the child needs to at least look back and see the result of the image making act as a whole(and in fact this may entail making a drawing or writing a symbolic phrase or sentence.)

We can now point out several consequences - or theorems - of our theory for the observation of understanding and its growth. Although process and form are complementary at every level, we envisage growth from level to level as occuring through a process which is then completed with a form to create a new level. One cannot build a form at a new level without first having a process at that level. In our model the process part of a level is considered to lead on to the form for that level. By the same token a person would not be observed to have fully attained a particular level of understanding unless they have exhibited the form-oriented actions of that level such as re-viewing or property recording. Each process segment can be seen to enfold all previous levels and to act on them in some fashion. If the process aspect gives the "continuity" with inner levels of knowing, it is the form which gives distinct substance to each level.

Obviously movement through, or within, this model is not automatic, and we will consider the role of the teacher in sponsoring such growth and movement later in this essay. But we would like to make a couple of other observations first. It is not essential for a person to have a complete knowledge of a topic at any particular level in order to move outwards. Quite the opposite is true. One moves outwards until something triggers the need to fold back, in order to alter or add to ones previous image, property list, or formalisation. Folding back can be to any inner level, not necessarily to the immediately preceding level. Indeed, the form/ process complementarity allows one to fold back to either the form or the process at any level with the purpose of informing understanding at some outer level. This can happen directly or through re-tracing the path of growth through the intervening levels. The nature of the path of this folding back, however, cannot be generally prescribed; it is uniquely drawable only for specific examples of growth in thinking such as that exemplified earlier, for the addition of fractions. Each person would have a unique path for any topic. Yet all paths would involve 'folding back to move out' in their actualisation.

At this point we turn to the question of the value of such a theory. With Einstein (quoted in Fine, 1986) we see a

theory as: "a self-sharpening tool whose warrants and value in the end rest on this, that they permit the coordination of experience, 'with dividends' [mit vorteil]." We hope that we have given some indication of the kind of dividends such a theory might provide.

We see significant relationships between this theory and the practice of mathematics teaching. We think it can be used to observe and inform such practice on a variety of temporal and conceptual scales. We have used it (Pirie and Kieren, 1989; Kieren and Pirie [forthcoming]) as a tool to observe the mathematical behaviour of children as they work on a single mathematical task and as they build and organise mathematical knowledge structures over periods of time. It can also be used to observe and to provide a frame for planning and executing mathematics lessons. In addition we have used it to make observations about curriculum developement (Kieren - in collaboration with Pirie - forthcoming).

What is it that a teacher can do to promote growth and understanding ? One consequence of the theory is very clear - <u>teachers cannot give a student understanding.</u> Only the student can create her own understanding. The role of the teacher is to provoke and enable growth. This was illustrated in our earlier example of "algorithm versus folding back" in the addition of fractions.

Simply asking students to engage in what appears to be high level mathematics is not to be mistaken with providing them with deep understanding. Giving a student a theorem and a proof, and then having the student mimic this proof on closely related questions should not be considered as necessarily providing deep understanding. Giving proofs for students to work on, could engage them in making an image of mathematical proof, if they have the appropriate back-ground of primitive doing, and thus could lead to <u>genuine</u> deeper understanding.

Teachers need to make opportunities to promote understanding. We see three kinds of 'questioning' as crucial to this task. (questioning is used here to mean more than mere interogation.) We have called these pro-vocative, in-vocative, and validating questions. We will simply give you brief definitions here as this is an area we are currently exploring. Pro-vocative questions are those which point a student towards an outer layer. In-vocative questions are those which cause a student to see the need to fold back to an earlier level in order to progress. It would be quite possible for a teacher to pose what she hoped would be a pro-vocative question only to find that in fact the current state of the student's understanding caused that student to

interpret it in-vocatively. Thirdly, validating questions are, in simple terms, those that ask the student " why ?".

Summary

The purpose of this brief essay has been to show a trancendently recursive theory of mathematical understanding which is based on the consideration of understanding as a non-linear process of growth. This theory of growth has a built-in dynamic of folding back to move out, but also shows understanding as an organizing process to entail action, perception of action, conception of percepts, conception of concepts,conception of forms(or classes)and logical explanation in an embedded manner.

Does the theory show understanding to be a non-linear phenomonon? We view understanding as a whole entailing at least one level beyond primitive doing for a person working within an observed conceptual area and possibly containing all eight levels. Understanding is knowing at many levels at once. It is not a single thing or a linear combination thereof. Understanding can be observed as a path or a movement through these levels of knowing; but it is not observed as monotonically outwardly moving phenomenon. It is non-linear as such because it involves folding back to re-member and to re-construct.

In what sense is the theory recursive? We would argue that it is a levelled whole such that it maintains self similarity with level stepping. It can be shown to have the recursive property of using similar but transcendent processes on new mathematical phenomena and conversely to use outer level processes to rebuild inner level knowing.

In what sense is the theory transcendent? It involves outer ways of knowing which are compatible with their inner levels but which are not reducible to the inner levels. Further, outer levels involve new forms, the treatment of new kinds of mathematical objects and ideas, and the use of different kinds of arguements in justifying such knowing and understanding.Understanding is a transcendent recursive whole.

We'll end with a favourite quote from Maturana:

> At this point there is either much more to say
> or nothing.

When we first said it we were not sure which of these options obtained. We hope we have demonstrated that there is much more to say.

References

Einstein A. (as quoted in Fine, A.)(1986). The Shakey Game- Einstein Realism and the Quantum Theory. Chicago: U. of Chicago Press, p. 91

Kieren, T. E.(in collaboration with S. E. B. Pirie) (in press) Rational Numbers: From Quotient Fields to Recursive Understanding. In T. Carpenter and L. Fennema(eds.) Cognitively Guided Instruction for

Kieren, T. and S. Pirie (Forthcoming) Recursion and the Mathematical Experience. In L. Steffe (ed) The Springer Verlag Psychology Series.

Margenau, H.(1987) The Miracle of Existence. Boston, New Science Library, Shambhala

Maturana, H.R. and F.J. Varela (1980) Autopoeisis and Cognition. Boston University, Phil. of Sci. Series, Vol. 42, Dordrecht, D. Riedel.

Maturana, H.R. and F.J. Varela (1987) The Tree of Knowledge. Boston : The New Sciences Library, Shambhala
Minsky, M.(1985) Society of the Mind. New York: Viking.

National Council of Teachers of Mathematics Commission on Standards for School Mathematics (1989) Curriculum and Evaluation Standards for School Mathematics. Reston: NCTM

Piaget, J.(1980) Adaptation and Intelligence:Organic Selection and Phenocopy. Chicago: U. of Chicago Press

Pirie, S.E.B. (1988) "Understanding - Instrumental, Relational, Formal, Intuitive . . . , How Can We Know." in For the Learning of Mathematics. Vol. 8,No.3, pp 2-6.

Pirie, S.E.B., and T. Kieren(1989) A Recursive Theory of Mathematical Understanding . In For The Learning of Mathematics, Vol. 9, No. 3, Nov.

Schroder, T.L. (1987) Students' Understanding of Mathematics: A Review and Synthesis of Some Recent Research. In J. Bergeron, N. Herscovics, and C. Kieran(eds.) Psychology of Mathematics Education XI, Vol.3. Montreal: PME

Skemp, R. R.(1987) The Psychology of Learning Mathematics, expanded American edition. Hillsdale, N.J.: Lawrence Erlbaum Associates.

Tall, D. (1978) The Dynamics of Understanding Mathematics. In Mathematics Teaching, 84,pp 50-52

Tomm, K. (1989) Consciousness and Intentionality in the Work of Humberto Maturana, a presentation for the Faculty of Education, University of Alberta.

von Glasersfeld, E. (1987) Learning as a Constructive Activity. In C. Janvier(ed) Problems of Representation in the Learning and Teaching of Mathematics. Hillsdale, N.J.:Lawrence Erlbaum Associates.

Vitale, B. (1989) Elusive Recursion: A Trip in a Recursive Land in New Ideas in Psychology, November.